

ON THE MOST EFFICIENT UNITARY TRANSFORMATION FOR PROGRAMMING QUANTUM CHANNELS*

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We address the problem of finding the optimal joint unitary transformation on system + ancilla which is the most efficient in programming any desired channel on the system by changing the state of the ancilla. We present a solution to the problem for $\dim(H) = 2$ for both system and ancilla.

Keywords: Quantum information theory; channels; quantum computing; entanglement

1. Introduction

A fundamental problem in quantum computing and, more generally, in quantum information processing¹ is to experimentally achieve any theoretically designed quantum channel with a fixed device, being able to program the channel on the state of an ancilla. This problem is of relevance for example in proving the equivalence of cryptographic protocols, e. g. proving the equivalence between a multi-round and a single-round quantum bit commitment². What makes the problem of channel programmability non trivial is that exact universal programmability of channels is impossible, as a consequence of a no-go theorem for programmability of unitary transformations by Nielsen and Chuang³. A similar situation occurs for

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universal programmability of POVM's^{4,5}. It is still possible to achieve programmability probabilistically⁶, or even deterministically⁷, though within some accuracy. Then, for the deterministic case, the problem is to determine the most efficient programmability, namely the optimal dimension of the program-ancilla for given accuracy. Recently, it has been shown⁵ that a dimension increasing polynomially with precision is possible: however, even though this is a dramatical improvement compared to preliminary indications of an exponential grow⁸, still it is not optimal.

In establishing the theoretical limits to state-programmability of channels and POVM's the starting problem is to find the joint system-ancilla unitary which achieves the best accuracy for fixed dimension of the ancilla: this is exactly the problem that is addressed in the present paper. The problem turned out to be hard, even for low dimension, and here we will give a solution for the qubit case, for both system and ancilla.

2. Statement of the problem

We want to program the channel by a fixed device as follows

$$\mathcal{P}_{V,\sigma}(\rho) \doteq \text{Tr}_2[V(\rho \otimes \sigma)V^\dagger], \quad (1)$$

with the system in the state ρ interacting with an ancilla in the state σ via the unitary operator V of the programmable device (the state of the ancilla is the *program*). For fixed V the above map can be regarded as a linear map from the convex set of the ancilla states \mathcal{A} to the convex set of channels for the system \mathcal{C} . We will denote by $\mathcal{P}_{V,\mathcal{A}}$ the image of the ancilla states \mathcal{A} under such linear map: these are the programmable channels. According to the well known no-go theorem by Nielsen and Chuang it is impossible to program all unitary channels on the system with a single V and a finite-dimensional ancilla, namely the image convex $\mathcal{P}_{V,\mathcal{A}} \subset \mathcal{C}$ is a proper subset of the whole convex \mathcal{C} of channels. This opens the following problem:

Problem: *For given dimension of the ancilla, find the unitary operators V that are the most efficient in programming channels, namely which minimize the largest distance $\varepsilon(V)$ of each channel $\mathcal{C} \in \mathcal{C}$ from the programmable set $\mathcal{P}_{V,\mathcal{A}}$:*

$$\varepsilon(V) \doteq \max_{\mathcal{C} \in \mathcal{C}} \min_{\mathcal{P} \in \mathcal{P}_{V,\mathcal{A}}} \delta(\mathcal{C}, \mathcal{P}) \equiv \max_{\mathcal{C} \in \mathcal{C}} \min_{\sigma \in \mathcal{A}} \delta(\mathcal{C}, \mathcal{P}_{V,\sigma}). \quad (2)$$

As a definition of distance it would be most appropriate to use the CB-norm distance $\|\mathcal{C} - \mathcal{P}\|_{CB}$. However, this leads to a very hard problem. We

will use instead the following distance

$$\delta(\mathcal{C}, \mathcal{P}) \doteq \sqrt{1 - F(\mathcal{C}, \mathcal{P})}, \quad (3)$$

where $F(\mathcal{C}, \mathcal{P})$ denotes the Raginsky fidelity⁹, which for unitary map $\mathcal{C} \equiv \mathcal{U} = U \cdot U^\dagger$ is equivalent to the channel fidelity¹

$$F(\mathcal{U}, \mathcal{P}) = \frac{1}{d^2} \sum_i |\text{Tr}[C_i^\dagger U]|^2, \quad (4)$$

where $\mathcal{C} = \sum_i C_i \cdot C_i^\dagger$. Such fidelity is also related to the input-output fidelity averaged over all pure states $\bar{F}_{io}(\mathcal{U}, \mathcal{P})$, by the formula $\bar{F}_{io}(\mathcal{U}, \mathcal{P}) = [1 + dF(\mathcal{U}, \mathcal{P})]/(d+1)$. Therefore, our optimal unitary V will maximize the fidelity

$$F(V) \doteq \min_{U \in \mathbf{U}(\mathbf{H})} F(U, V), \quad F(U, V) \doteq \max_{\sigma \in \mathcal{A}} F(\mathcal{U}, \mathcal{P}_{V, \sigma}) \quad (5)$$

3. Reducing the problem to an operator norm

In the following we will use the GNS representation $|\Psi\rangle\rangle = (\Psi \otimes I)|I\rangle\rangle$ of operators $\Psi \in \mathbf{B}(\mathbf{H})$, and denote by X^\top the transposed with respect to the cyclic vector $|I\rangle\rangle$, i. e. $|\Psi\rangle\rangle = (\Psi \otimes I)|I\rangle\rangle = (I \otimes \Psi^\top)|I\rangle\rangle$, and by X^* the complex conjugated operator $X^* \doteq (X^\top)^\dagger$, and write $|v^*\rangle$ for the vector such that $(|v\rangle\langle v| \otimes I)|I\rangle\rangle = |v\rangle|v^*\rangle$. Upon spectralizing the unitary V as follows

$$V = \sum_k e^{i\theta_k} |\Psi_k\rangle\rangle \langle\langle \Psi_k|, \quad (6)$$

we obtain the Kraus operators for the map $\mathcal{P}_{V, \sigma}(\rho)$

$$\mathcal{P}_{V, \sigma}(\rho) = \sum_{nm} C_{nm} \rho C_{nm}^\dagger, \quad C_{nm} = \sum_k e^{i\theta_k} \Psi_k |v_n^*\rangle \langle v_m^*| \Psi_k^\dagger \sqrt{\lambda_m} \quad (7)$$

where $|v_n\rangle$ denotes the eigenvector of σ corresponding to the eigenvalue λ_n . We then obtain

$$\begin{aligned} \sum_{nm} |\text{Tr}[C_{nm}^\dagger U]|^2 &= \sum_{kh} e^{i(\theta_k - \theta_h)} \text{Tr}[\Psi_k^\dagger U^\dagger \Psi_k \sigma^\top \Psi_h^\dagger U \Psi_h] \\ &= \text{Tr}[\sigma^\top S(U, V)^\dagger S(U, V)] \end{aligned} \quad (8)$$

where

$$S(U, V) = \sum_k e^{-i\theta_k} \Psi_k^\dagger U \Psi_k. \quad (9)$$

The fidelity (5) can then be rewritten as follows

$$F(U, V) = \frac{1}{d^2} \|S(U, V)\|^2. \quad (10)$$

4. Solution for the qubit case

The operator $S(U, V)$ in Eq. (9) can be written as follows

$$S(U, V) = \text{Tr}_1[(U^\top \otimes I)V^*]. \quad (11)$$

Changing V by local unitary operators transforms $S(U, V)$ in the following fashion

$$S(U, (W_1 \otimes W_2)V(W_3 \otimes W_4)) = W_2^* S(W_1^\dagger U W_3^\dagger, V) W_4^*, \quad (12)$$

namely the local unitaries do not change the minimum fidelity, since the unitaries on the ancilla just imply a different program state, whereas the unitaries on the system just imply that the minimum fidelity is achieved for a different unitary—say $W_1^\dagger U W_3^\dagger$ instead of U .

For system and ancilla both two-dimensional, one can parameterize all possible joint unitary operators as follows¹⁰

$$V = (W_1 \otimes W_2) \exp[i(\alpha_1 \sigma_1 \otimes \sigma_1^\top + \alpha_2 \sigma_2 \otimes \sigma_2^\top + \alpha_3 \sigma_3 \otimes \sigma_3^\top)](W_3 \otimes W_4). \quad (13)$$

A possible quantum circuit to achieve V in Eq. (13) can be designed using the identities

$$\begin{aligned} [\sigma_\alpha \otimes \sigma_\alpha, \sigma_\beta \otimes \sigma_\beta] &= 0, \\ C(\sigma_x \otimes I)C &= \sigma_x \otimes \sigma_x, \\ C(I \otimes \sigma_z)C &= -\sigma_z \otimes \sigma_z, \\ \left(e^{-\frac{i\pi}{4}\sigma_z} \otimes e^{-\frac{i\pi}{4}\sigma_z}\right) C(\sigma_x \otimes I)C \left(e^{\frac{i\pi}{4}\sigma_z} \otimes e^{\frac{i\pi}{4}\sigma_z}\right) &= \sigma_y \otimes \sigma_y, \end{aligned} \quad (14)$$

where C denotes the controlled-NOT

$$C = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes \sigma_x. \quad (15)$$

This gives the quantum circuit in Fig. 1. The problem is now reduced to

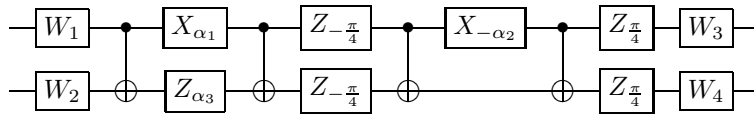


Figure 1. Quantum circuit scheme for the general joint unitary operator V in Eq. (13). Here we use the notation $G_\phi = \exp(i\phi\sigma_G)$ with $G = X, Y, Z$.

study only joint unitary operators of the form

$$V = \exp[i(\alpha_1 \sigma_1 \otimes \sigma_1^\top + \alpha_2 \sigma_2 \otimes \sigma_2^\top + \alpha_3 \sigma_3 \otimes \sigma_3^\top)]. \quad (16)$$

This has eigenvectors

$$|\Psi_j\rangle = \frac{1}{\sqrt{2}}|\sigma_j\rangle, \quad (17)$$

where σ_j , $j = 0, 1, 2, 3$ denote the Pauli matrices $\sigma_0 = I$, $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, $\sigma_3 = \sigma_z$. This means that we can rewrite $S(U, V)$ in Eq. (9) as follows

$$S(U, V) = \frac{1}{2} \sum_{j=0}^3 e^{-i\theta_j} \sigma_j U \sigma_j, \quad (18)$$

with

$$\theta_0 = \alpha_1 + \alpha_2 + \alpha_3, \quad \theta_i = 2\alpha_i - \theta_0. \quad (19)$$

The unitary U belongs to $\mathbb{SU}(2)$, and can be written in the Bloch form

$$U = n_0 I + i \mathbf{n} \cdot \boldsymbol{\sigma}, \quad (20)$$

with $n_k \in \mathbb{R}$ and $n_0^2 + |\mathbf{n}|^2 = 1$. Using the identity

$$\sigma_j \sigma_l \sigma_j = \epsilon_{jl} \sigma_l, \quad \epsilon_{j0} = \epsilon_{jj} = 1, \quad \epsilon_{jl} = -1, l \neq 0, j, \quad (21)$$

we can rewrite

$$S(U, V) = \tilde{n}_0 I + \tilde{\mathbf{n}} \cdot \boldsymbol{\sigma}, \quad (22)$$

where

$$\tilde{n}_j = t_j n_j, \quad 0 \leq j \leq 3, t_0 = \frac{1}{2} \sum_{j=0}^3 e^{-i\theta_j}, \quad (23)$$

$$t_j = e^{-i\theta_0} + e^{-i\theta_j} - t_0, \quad 1 \leq j \leq 3, \quad t_j = |t_j| e^{i\phi_j}, \quad 0 \leq j \leq 3,$$

It is now easy to evaluate the operator $S(U, V)^\dagger S(U, V)$. One has

$$S(U, V)^\dagger S(U, V) = v_0 I + \mathbf{v} \cdot \boldsymbol{\sigma}, \quad (24)$$

$$v_0 = |\tilde{n}_0|^2 + |\tilde{\mathbf{n}}|^2, \quad \mathbf{v} = i [2\Im(\tilde{n}_0 \tilde{\mathbf{n}}^*) + \tilde{\mathbf{n}}^* \times \tilde{\mathbf{n}}].$$

Now, the maximum eigenvalue of $S(U, V)^\dagger S(U, V)$ is $v_0 + |\mathbf{v}|$, and one has

$$|\mathbf{v}|^2 = \sum_{i,j=0}^3 |\tilde{n}_i|^2 |\tilde{n}_j|^2 - \tilde{n}_i^{*2} \tilde{n}_j^2 = 2 \sum_{i,j=0}^3 |\tilde{n}_i|^2 |\tilde{n}_j|^2 \sin^2(\phi_i - \phi_j), \quad (25)$$

whence the norm of $S(U, V)$ is given by

$$\|S(U, V)\|^2 = \sum_{j=0}^3 n_j^2 |t_j|^2 + \sqrt{2 \sum_{i,j=0}^3 n_i^2 n_j^2 |t_i|^2 |t_j|^2 \sin^2(\phi_i - \phi_j)}. \quad (26)$$

Notice that the unitary U which is programmed with minimum fidelity in general will not be unique, since the expression for the fidelity depends on $\{n_j^2\}$. Notice also that using the decomposition in Eq. (13) the minimum fidelity just depends on the phases $\{\theta_j\}$, and the local unitaries will appear only in the definitions of the optimal program state and of the worstly approximated unitary. It is convenient to write Eq. (26) as follows

$$\|S(U, V)\|^2 = \mathbf{u} \cdot \mathbf{t} + \sqrt{\mathbf{u} \cdot \mathbf{T}\mathbf{u}}. \quad (27)$$

where $\mathbf{u} = (n_0^2, n_1^2, n_2^2, n_3^2)$, $\mathbf{t} = (|t_0|^2, |t_1|^2, |t_2|^2, |t_3|^2)$, and $\mathbf{T}_{ij} = |t_i|^2 |t_j|^2 \sin^2(\phi_i - \phi_j)$. One has the bounds

$$\mathbf{u} \cdot \mathbf{t} + \sqrt{\mathbf{u} \cdot \mathbf{T}\mathbf{u}} \geq \mathbf{u} \cdot \mathbf{t} \geq \min_j |t_j|^2, \quad (28)$$

and the bound is achieved on one of the extremal points $u_i = \delta_{ij}$ of the domain of \mathbf{u} which is the convex set $\{\mathbf{u}, u_j \geq 0, \sum_j u_j = 1\}$ (the positive octant of the unit four dimensional ball S_+^4). Therefore, the fidelity minimized over all unitaries is given by

$$F(V) = \frac{1}{d^2} \min_j |t_j|^2. \quad (29)$$

The optimal unitary V is now obtained by maximizing $F(V)$. We need then to consider the decomposition Eq. (13), and then to maximize the minimum among the four eigenvalues of $S(U, V)^\dagger S(U, V)$. Notice that $t_j = \sum_\mu H_{j\mu} e^{i\theta_\mu}$, where H is the Hadamard matrix

$$H = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad (30)$$

which is unitary, and consequently $\sum_j |t_j|^2 = \sum_j |e^{i\theta_j}|^2 = 4$. This implies that $\min_j |t_j| \leq 1$. We now provide a choice of phases θ_j such that $|t_j| = 1$ for all j , achieving the maximum fidelity allowed. For instance, we can take $\theta_0 = 0, \theta_1 = \pi/2, \theta_2 = \pi, \theta_3 = \pi/2$, corresponding to the eigenvalues $i, 1, -i, 1$ for V . Another solution is $\theta_0 = 0, \theta_1 = -\pi/2, \theta_2 = \pi, \theta_3 = -\pi/2$. Also one can set $\theta_i \rightarrow -\theta_i$. The eigenvalues of $S(U, V)^\dagger S(U, V)$ are then $1, 1, 1, 1$, while for the fidelity we have

$$F \doteq \max_{V \in \mathcal{U}(\mathbb{H}^{\otimes 2})} F(V) = \frac{1}{d^2} = \frac{1}{4}, \quad (31)$$

and the corresponding optimal V has the form

$$V = \exp \left[\pm i \frac{\pi}{4} (\sigma_x \otimes \sigma_x \pm \sigma_z \otimes \sigma_z) \right]. \quad (32)$$

A possible circuit scheme for the optimal V is given in Fig. 2.

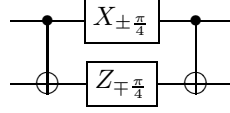


Figure 2. Quantum circuit scheme for the optimal unitary operator V in Eq. (31). For the notation see Fig. 1. For the derivation of the circuit see Eqs. (14).

We now show that such fidelity cannot be achieved by any V of the controlled-unitary form

$$V = \sum_{k=1}^2 V_k \otimes |\psi_k\rangle\langle\psi_k|, \quad \langle\psi_1|\psi_2\rangle = 0, \quad V_1, V_2 \text{ unitary on } \mathcal{H} \simeq \mathbb{C}^2. \quad (33)$$

For spectral decomposition $V_k = \sum_{j=1}^2 e^{i\theta_k^{(j)}} |\phi_j^{(k)}\rangle\langle\phi_j^{(k)}|$ the eigenvectors of V are $|\Psi_{jk}\rangle = |\phi_j^{(k)}\rangle|\psi_k\rangle$, and the corresponding operators are $\Psi_{jk} = |\phi_j^{(k)}\rangle\langle\psi_k^*|$, namely the operator $S(U, V)$ is

$$S(U, V) = \sum_{j,k} e^{-i\theta_k^{(j)}} |\psi_k^*\rangle\langle\phi_j^{(k)}| U |\phi_j^{(k)}\rangle\langle\psi_k^*|, \quad (34)$$

with singular values $\sum_{j=1}^2 e^{-i\theta_k^{(j)}} \langle\phi_j^{(k)}|U|\phi_j^{(k)}\rangle = \text{Tr}[V_k^\dagger U]$. Then, the optimal program state is $|\psi_h\rangle$, with $h = \arg \max_k |\text{Tr}[V_k^\dagger U]|$, and the corresponding fidelity is

$$F(U, V) = \frac{1}{4} |\text{Tr}[V_h^\dagger U]|^2, \quad (35)$$

and one has

$$F(V) = \min_U F(U, V) = 0, \quad (36)$$

since for any couple of unitaries V_k there always exists a unitary U such that $\text{Tr}[V_k^\dagger U] = 0$ for $k = 1, 2$. Indeed, writing the unitaries in the Bloch form (20), their Hilbert-Schmidt scalar is equal to the euclidean scalar product in \mathbb{R}^4 of their corresponding vectors, whence it is always possible to find a vector orthogonal to any given couple in \mathbb{R}^4 . The corresponding U is then orthogonal to both V_k , and the minimum fidelity for any controlled-unitary is zero.

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